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USE OF THE REYNOLDS FLUX CONCEPT FOR ANALYSING ONE-DIMENSIONAL TWO-PHASE FLOW

PART I. DERIVATION AND VERIFICATION OF BASIC ANALYTICAL TECHNIQUES

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Abstract—This series of two papers has the following objective: (1) To draw attention to the usefulness of the Reynolds flux concept in engineering analysis, to compare the various theories of previous workers, and to present some novel developments. (2) To show how the concept can be applied to predict pressure gradients in forced convection boiling, flashing and condensation. This analysis is also related to the mechanisms of entropy production in two-phase flow.

NOMENCLATURE

B,	Spalding parameter equation (56);
c,	concentration;
c_a,	concentration of non-condensables;
c_p,	specific heat at constant pressure;
f,	friction factor;
h,	heat-transfer coefficient;
h_D,	mass-transfer coefficient;
M,	molecular weight;
$\dot{m}$,	mass flux;
m,	mass flux due to evaporation, condensation, injection or suction;
$p(u)$,	probability of velocity (u);
P,	pressure or partial pressure;
$\dot{q}$,	heat flux;
R,	universal gas constant;
T,	temperature;
u,	normal velocity characterizing mixing;
V,	main stream velocity;
v,	mean normal velocity due to mass transfer;
v',	normal velocity imposed on mixing pattern due to mass transfer;
X,	any conserved property;
Pr,	Prandtl number;
Sc,	Schmidt number.

Greek symbols

β,	fraction of area occupied by steam approaching the wall;
ε,	Reynolds flux;
ε_1,	Reynolds flux towards wall or interface;
ε_2,	Reynolds flux away from wall or interface;
ε_m,	mixing flux in the presence of mass transfer;
η,	dimensionless parameter;
ρ,	density;
τ_w,	wall shear stress.

Subscripts

a, b,	components a and b;
c,	concentration;
e,	energy;
m,	with mass injection, or property of injected stream;
o,	without mass injection;
p,	momentum;
s,	main stream;
w,	wall.

The prefix Δ denotes a definite difference (e.g. ΔT Δc).

INTRODUCTION

MODERN engineering analysis usually starts from some very general scientific laws and is developed in a deductive, logical and consistent mathematical sequence. This is in contrast with the traditional approach in which some simple special cases were first examined, usually in conjunction with experimental data, and were used as a basis for successive developments to more sophisticated levels of analysis. The justification for the newer method is that it reduces the risk of neglecting significant effects in an attempt to force a problem to conform to a particular simplified model which may be inappropriate. However, the danger of the general approach is that its disciples are often quite helpless when confronted with "complicated" situations in which the equations give rise to enormous mathematical difficulties. An emphasis on mathematical rigor may well tend to suppress the very flights of imagination which can lead to the way out of the difficulty. The present status of theories of turbulence is perhaps an example of such a situation. Even more typical of this class of problems are those in which heat, mass and momentum transfer occur simultaneously in systems of complex geometry when several phases are present in a variety of flow patterns.

In these two papers an attempt will be made to show the utility of exploiting the simplest level of sophistication which will describe convective transfer phenomena. Some novel theoretical developments will be offered. In particular, the results will be used to develop new techniques, for handling problems of two-phase flow, boiling, flashing and condensation.

The basic ideas which will be used go back almost one hundred years to the work of Reynolds [1] who suggested a theory relating friction and heat transfer in turbulent flow. Half a century later a significant contribution was made by Nusselt [2] who extended the application of these concepts to combustion processes. In the last two decades Silver [3, 4, 5] has exploited the methods extensively and Spalding [6] has based an entire monograph on them. Applications to two-phase flow [7, 8, 9] are of very recent date. Moreover, some of the simplest developments which will be made in this paper, based on the superposition of lateral velocities, appear to have received no previous attention.

In essence the "Reynolds flux" concept postulates transverse mixing velocities in order to account for transport phenomena in a one-dimensional flow model. This simple-minded view is consistent with the level of analytical sophistication which is accorded to the main flow. Clearly it cannot be expected to compete in accuracy with more elaborate techniques. However, the results of these more exact calculations can usually be expressed in terms of the parameters which characterize the simpler model. These more accurate solutions for particular cases can then be presented as improvements over the more general, but approximate, results.

One may well wonder why the Reynolds flux has not received wider attention. It is often regarded as an "analogy" which is academically interesting but inaccurate. This attitude is opposed by the contention that the concept is both the logical outcome of choosing a lumped-parameter description of the flow and is also the macroscopic equivalent of the diffusion laws for describing irreversible processes. The interested reader will perhaps incline more towards the latter opinion after reading these papers.

Finally, since no scientific work is ever complete, it is hoped that parts of these papers will stimulate further developments in a fascinating area of research which is still chronically short of useful basic analytical techniques.

THEORY

The concept of a lateral mixing flux

The analysis of fluid flow is often based on a "one-dimensional" idealization in which the flow is assumed to be characterized by a uniform average velocity V. This method of analysis is generally satisfactory as long as the walls of the duct do not diverge significantly and lateral variations across the flow are not important.

Since the purely one-dimensional model cannot in any way account for heat, mass and momentum transfer normal to the direction of flow, some additional concept must be introduced if these phenomena are to be considered. It is well known that these effects can be described by a suitable definition of friction factor, f, heat-transfer coefficient, h, and mass-transfer coefficient h_D while retaining the one-dimensional (lumped-parameter) description of the flow. The wall shear stress, heat flux and mass transfer are then

$$\tau_w = f/2\,\rho V^2 \tag{1}$$

$$\dot{q} = h\,\Delta T \tag{2}$$

$$\dot{m} = h_D\,\Delta_c. \tag{3}$$

If there is a coupling between these processes or if the rate of mass transfer is large the equations are more complicated.

Equations (1–3) give operational definitions of f, h and h_D, whatever the detailed fluid mechanics may be.

Most authors introduce the Reynolds flux concept by making particular reference to turbulent flow. It is assumed that the mixing processes can be idealized and viewed in terms of a lateral mass flux, ε_o, which leaves the main flow, shares its momentum, energy and composition with a layer close to the wall and then bounces back into the main stream again. The relation which is often Reynolds "analogy" is then readily found to be

$$\varepsilon_o = \frac{f}{2}\rho V = \frac{h}{c_p} = \rho h_D. \tag{4}$$

Although this result is usually deduced from a very simple-minded model for turbulence it should be appreciated that the actual concept of a "lateral mixing flux" implies no distinct hydrodynamic mechanism. Since energy, momentum and species are ultimately all properties of the individual molecules, one might as well regard ε_o as resulting from a net molecular flux across the boundaries of a control surface, by whatever means may be appropriate.

An alternative approach to the Reynolds flux can be made from the point of view of transverse velocities in the flow [9]. Again, it does not matter whether these velocities are the result of molecular motion or of the flow of particles of fluid. The simplest approach is to postulate transverse average velocities of magnitude u_o. On the average one half of the transverse flow will be approaching the wall with velocity u_o and one half will be leaving with this velocity. An idealized representation of this state of affairs is shown in Fig. 1. The relationship between ε_o and u_o is

$$\varepsilon_o = \tfrac{1}{2}\rho u_o. \tag{5}$$

Substituting (5) into (4) we obtain

$$u_o = fV = \frac{2h}{\rho c_p} = 2h_D. \tag{6}$$

The friction factor could therefore be regarded as the ratio between the lateral fluctuating velocity and the main stream velocity.

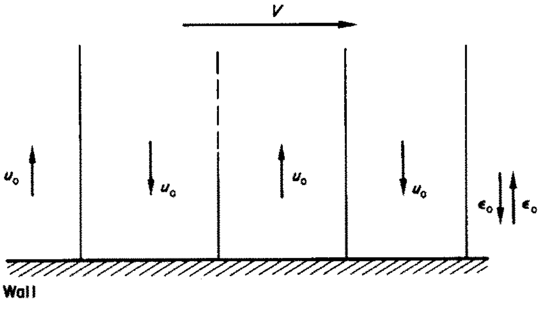

FIG. 1. Idealized transverse velocity model of the Reynolds flux. $\varepsilon_o = \frac{1}{2}\rho u_o$.

It is quite possible that under some circumstances the mixing pattern may be "lop-sided" with unequal velocities towards and away from the wall. If the fraction of the transverse area which is occupied by the stream which approaches the wall is β, then the appropriate mixing velocities are

$$u_o/2\beta \qquad \text{towards the wall,}$$

$$u_o/2(1-\beta) \text{ away from the wall.}$$

Alternatively there may be a stagnant area which does not participate in the mixing process, in which case both transverse velocities could be increased by the factor $1/\beta$.

Equations (4) and (6) define what is apparently the simplest analytical model which will account for transfer phenomena in convection. An improved empirical modification which introduces no greater level of sophistication but accounts for the different rates of transport of mass, momentum and energy is due to Chilton and Colburn [10] in the well-known form

$$u_o = fV = \frac{2h}{\rho c_p} Pr^{\frac{2}{3}} = 2h_D Sc^{\frac{2}{3}}. \tag{7}$$

Equation (7) is approximately valid for both laminar and turbulent boundary layers and for developed turbulent flow except where form drag is appreciable or when the Prandtl and Schmidt numbers are very far from unity.

Another approach at this point is to define separate transverse velocities for the three processes, such that the simple model is valid for each. Using subscripts p, e and c to denote momentum, energy and composition we then have

$$(u_p)_o = fV \tag{8}$$

$$(u_e)_o = \frac{2h}{\rho c_p} \tag{9}$$

$$(u_c)_o = 2h_D. \tag{10}$$

It follows from (7) that these velocities are related by the equation

$$(u_p)_o = (u_e)_o Pr^{\frac{2}{3}} = (u_c)_o Sc^{\frac{2}{3}}. \tag{11}$$

Evidently, ascribing only one value to the transverse velocity for each process amounts to a considerable idealization. In fact a whole spectrum of transverse velocities will exist. In laminar flow they result from molecular motion, whereas in turbulent flow they involve the motion of macroscopic quantities of fluid. One way to account for these phenomena is to increase the level of sophistication and to consider two- and three-dimensional effects. A common practice is to use boundary-layer theory. The disadvantage of these more sophisticated techniques is that increased accuracy is bought at the expense of increased complexity. Specific solutions have to be generated for each change in geometry.

The alternative procedure is to retain the one-dimensional idealizations as long as possible while seeking ways to account for velocity fluctuations. If u_0 is regarded as the average or "expected" value of the transverse velocity, and it is admitted that all values of this velocity have some finite probability of occurring a least-prejudiced probability distribution can be generated, following the methods of Jaynes [11] and Tribus [12] and is found to be [9]

$$p(u)\,\mathrm{d}u = \frac{1}{u_o}\exp\left(-u/u_o\right)\mathrm{d}u. \tag{12}$$

In this equation "u" is to be interpreted as the magnitude of the velocity, since there is no reason for assuming that the inwards and outwards fluxes are in any way different.

We note at this point that the result which was obtained in equation (12) contains a subjective element. If we had chosen, for example, to characterize the fluctuating velocities with some mean energy, the result would have been a probability distribution proportional to $\exp\left[-\eta u^2/(u^2)_o\right]$ where $(u^2)_o$ represented the mean of the velocity squared. This latter model is consistent with the kinetic theory of gases. In the present instance, however, the average transverse velocity is the basic quantity, both from the conceptual and the experimental viewpoint.

Perhaps equation (12) could be regarded as describing the most "random" transfer process whereas the most orderly process would only allow values of $+u_o$ and $-u_o$, each with probability one half.

The choice of a probability distribution, such as is given by equation (12), makes no difference to equations (6) and (7) but will be found to be of importance when considering the effect of net mass transfer, suction, injection and phase change.

The effect of net mass transfer

One might expect that there would be some interaction between any directional mass transfer flux "m" and the fluctuating components of the Reynolds flux. Assuming, for the moment, that the material which is being transferred is the same as that which is in the main stream, we can define a transverse velocity, v, by the equation

$$m = \rho v. \tag{13}$$

For example, if gas were continuously injected through a porous wall at a rate m per unit time, its average velocity on leaving the wall would be v. If the mixing process, which is characterized by the velocity u, is unchanged in its structure by the presence of net mass transfer, then the stream which approaches the wall will acquire a speed $u - v$ and that which leaves will have a corresponding speed $u + v$, as shown in Fig. 2.

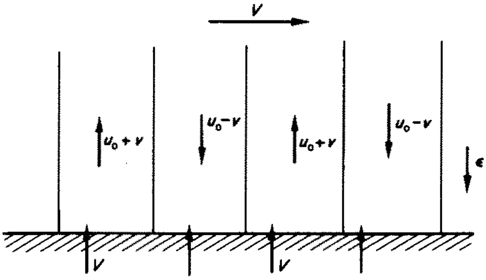

FIG. 2. Superposition of a velocity, v, due to injection (or evaporation) on the transverse velocities, u_o. $\varepsilon_1 = \frac{1}{2}\rho(u_o - v)$; $\varepsilon_2 = \frac{1}{2}\rho(u_o + v)$.

In the simplest model in which u is put equal to u_0 it is readily seen that the flux of matter which travels from the stream to the wall is reduced to the value

$$\varepsilon_1 = \frac{1}{2}\rho(u_o - v) \tag{14}$$

whereas the total rate of mass transport from the wall to the stream is now

$$\varepsilon_2 = \tfrac{1}{2}\rho(u_o + v). \tag{15}$$

Substituting for u_o and v from (13) and (5) into (14) and (15) we find

$$\varepsilon_1 = \varepsilon_o - m/2 \tag{16}$$

$$\varepsilon_2 = \varepsilon_o + m/2. \tag{17}$$

In the case of a "lop-sided" mixing pattern the equivalent results are

$$\varepsilon_1 = \varepsilon_o - \beta m \tag{18}$$

$$\varepsilon_2 = \varepsilon_o + (1 - \beta)m. \tag{19}$$

The above results are equally valid for positive or negative values of m as long as the algebraic sign conventions are preserved.

Using the same analytical model but employing the probability distribution given by equation (12) and assuming a symmetrical mixing pattern we obtain[9]

$$\varepsilon_1 = \tfrac{1}{2}\rho \int_v^\infty (u - v)\, p(u)\, \mathrm{d}u \tag{20}$$

or

$$\varepsilon_1 = \rho \int_v^\infty \frac{u - v}{2u_o} \exp(-u/u_o)\, \mathrm{d}u. \tag{21}$$

Performing the integration leads to the result

$$\varepsilon_1 = \rho \frac{u_o}{2} \exp(-v/u_o). \tag{22}$$

Similarly, for the total outgoing flux we have

$$\varepsilon_2 = \rho \int_0^v \frac{u - v}{2u_o} \exp(-u/u_o)\, \mathrm{d}u + \rho \int_0^\infty \frac{u + v}{2u_o} \exp(-u/u_o)\, \mathrm{d}u \tag{23}$$

which gives, on integration

$$\varepsilon_2 = \rho v + \rho \frac{u_o}{2} \exp(-v/u_o). \tag{24}$$

Substituting for u_o and v as before gives

$$\varepsilon_1 = \varepsilon_o \exp(m/2\varepsilon_o) \tag{25}$$

$$\varepsilon_2 = m + \varepsilon_o \exp(-m/2\varepsilon_o). \tag{26}$$

In this case one cannot simply reverse the sign of m in order to obtain the equivalent result for mass transfer towards the wall. The same conclusion is reached if one chooses other continuous probability distributions.

There are two paths which the analysis can now take. The first is to treat positive and negative values of m with the same basic model but to derive different equations for each case; the second is to seek a new level of conceptualization which will enable the powerful property of symmetry to be retained. The "best" choice will of course, depend on the nature of the particular problem which one wishes to solve.

Looking at the situation which is represented by Fig. 2 and the pairs of equations (16) and (17), (18) and (19) or (25) and (26), it is clear that the basic cause of asymmetry is that the flux "m" is unidirectional. If m is subtracted from ε_2 in each case the result is the same as ε_1. We can therefore regard the transport phenomena as made up of two superimposed processes—a "flow" and a "transfer". The flow is given by the flux m and transports matter, energy and molecular species irrespective of whatever the mixing process may be. The "transfer" is due to a mixing process characterized by a two-way flux which we may call ε_m. For a symmetrical mixing process we find, from (16) and (17), that

$$\varepsilon_m = \varepsilon_o - \frac{|m|}{2}. \tag{27}$$

It seems unlikely that ε_m can ever be negative, therefore for values of $|m/\varepsilon_o|$ greater than 2 we have

$$m > 2\varepsilon_o; \qquad \varepsilon_m = 0. \tag{28}$$

Alternatively, from (25) and (26)

$$\varepsilon_m = \varepsilon_o \exp(|m|/2\varepsilon_o). \tag{29}$$

In cases in which heat, mass and momentum transfer are characterized by different "mixing velocities" equations (27 and (28) become

$$(u_t)_m = (u_t)_o - |v| \tag{30}$$

or

$$(u_t)_m = (u_t)_o \exp[-|v|/(u_t)_o] \tag{31}$$

in which u_t denotes the appropriate transfer velocity (u_p, u_e or u_c for example).

A striking example of these results is that the mixing process is reduced to exactly the same degree by either positive or negative values of m. Thus, although the fluxes ε_1 and ε_2 are continuous functions as m passes through zero, ε_m passes through both a maximum and a discontinuity (Fig. 3). This point does not appear to have received adequate attention from workers in this field in the past.

For example, Fig. 3 shows the effect of suction on wall shear stress in a laminar boundary layer from the work of Iglisch [13]. The results are plotted in two ways. The shear stress is given, using the Reynolds flux model, by the expression

$$\tau_w = V\varepsilon_1 = V(\varepsilon_m + m). \tag{32}$$

The ratio between the shear stresses with and without suction is

$$\frac{\tau_w}{\tau_o} = \frac{\varepsilon_1}{\varepsilon_o} = \frac{\varepsilon_m + m}{\varepsilon_o}. \tag{33}$$

Using equation (29) we obtain

$$\frac{\varepsilon_1}{\varepsilon_o} = \frac{m}{\varepsilon_o} + \exp\left(-\frac{m}{2\varepsilon_o}\right) \tag{34}$$

whereas equation (27) gives

$$\begin{aligned} \frac{\varepsilon_1}{\varepsilon_o} &= 1 + \tfrac{1}{2}\frac{m}{\varepsilon_o} && \text{for} \quad m < 2\varepsilon_o \\ \frac{\varepsilon_1}{\varepsilon_o} &= \frac{m}{\varepsilon_o} && \text{for} \quad m > 2\varepsilon_o. \end{aligned} \tag{35}$$

Alternatively we may subtract m/ε_o from equation (33), plot the results in terms of $\varepsilon_m/\varepsilon_o$, and compare with equations (27) and (28) directly.

For comparison, some results of the effect of injection on wall shear stress in both laminar and turbulent boundary layers on a flat plate are shown in Fig. 4, which is reproduced from [14]. Also shown on this graph are the theoretical predictions of Spalding [equation (57)] and Kutateladze and Leontiev [equation (58)].

The effect of molecular weight

The very simple-minded analysis which is presented above can presumably be modified in many ways to account for the effects of Mach number, finite temperature differences, Prandtl

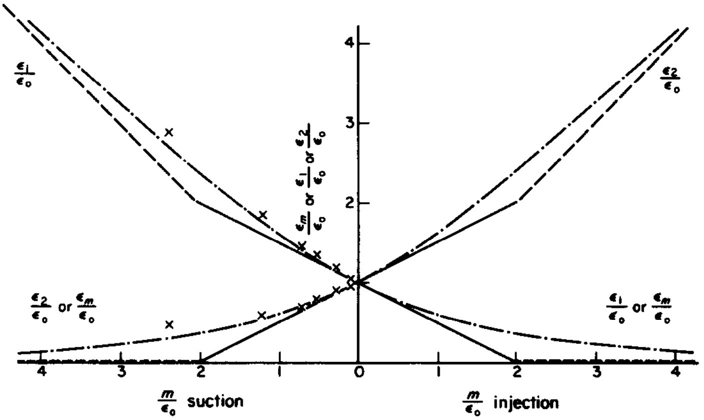

FIG. 3. The effect of suction or injection on the parameters $\varepsilon_1/\varepsilon_0$, $\varepsilon_2/\varepsilon_0$ and $\varepsilon_m/\varepsilon_0$.

× Results of Iglish [13];
——— Equations (16), (17), (27) and (35);
— — — Equations (28) and (35);
— · — · Equations (29) and (34).

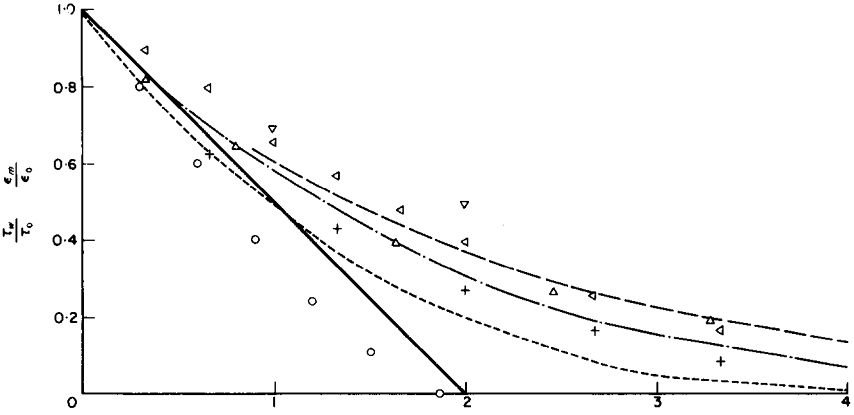

FIG. 4. Effect of injection on wall shear stress.

— — — Equation (29), ——— Equation (27),
- - - - - - - Equation (58), — · — Equation (57),
○ Hartnett and Eckert [20] laminar boundary layer, flat plate;
△ Hartnett and Eckert [20] laminar stagnation flow;
◁ Torii, Nishiwaki and Hirata [21] turbulent boundary layer, flat plate, Reynolds number = 10^5;
+ Torii, Nishiwaki and Hirata [21] turbulent boundary layer, flat plate, Reynolds number = 10^7;
▽ Olson and Eckert [22] injection into turbulent pipe flow.

and Schmidt numbers which differ from unity, chemical reactions and numerous other phenomena. For example, suppose that the gas which is being transferred or injected has a molecular weight which differs from the molecular weight of the main stream. Let the temperature and pressure be everywhere uniform. Then, if the perfect gas laws are obeyed, the density of the injected gas will be related to the density of the main flow by the ratio of the molecular weights, as follows

$$\frac{\rho_m}{\rho} = \frac{M_m}{M}. \tag{36}$$

Equation (13) now becomes

$$m = \rho \frac{M_m}{M} v. \tag{37}$$

If one assumes the same superposition of normal velocities as before, the following result is readily obtained,

$$\varepsilon_m = \varepsilon_o - \frac{1}{2} \frac{M}{M_m} |m|. \tag{38}$$

Equation (38) appears to be approximately valid for laminar flows and can be related to the accepted formulae for molecular diffusion by the methods which will later be given in "Example 2". In turbulent flows, however, it appears to be more accurate to consider momentum conservation normal to the wall. If the directional momentum flux which is given to the recirculating flow is put equal to the injected momentum flux we have

$$mv = \rho v'^2 \tag{39}$$

in which v' is now the velocity which is to be superimposed on the mixing pattern rather than v. Rearrangement of (39) gives

$$v' = v \sqrt{(\rho_m/\rho)}. \tag{40}$$

The recirculating mass flux is now

$$\varepsilon_m = \rho/2\,(u_o - v') = \varepsilon_o - \tfrac{1}{2} m \sqrt{(\rho/\rho_m)} \tag{41}$$

or, in terms of molecular weight ratio

$$\varepsilon_m = \varepsilon_o - \tfrac{1}{2} m \sqrt{(M/M_m)}. \tag{42}$$

For example, the shear stress on a wall through which mass injection is occurring is reduced to the value

$$\tau_m = V \varepsilon_m. \tag{43}$$

The ratio between this shear stress and the shear stress which would exist without injection is, from (42)

$$\frac{\tau_m}{\tau_o} = 1 - \tfrac{1}{2} \frac{m}{\varepsilon_o} \sqrt{(M/M_m)}. \tag{44}$$

Equation (44) is compared with the results of Pappas and Okuno [16], for a free stream Mach number of 0·3, in Fig. 5. Also shown is the prediction which would result from the probability distribution given in equation (12), namely

$$\frac{\tau_m}{\tau_o} = \exp\left[-\frac{1}{2}\frac{m}{\varepsilon_o}\sqrt{(M/M_m)}\right]. \tag{45}$$

Figure 6 shows the effect of molecular weight on the "critical velocity" at which the shear stress goes to zero. The ordinate is the ratio of the critical velocity to the critical velocity which is obtained with equal molecular weight. Also shown in this figure are the predictions of several different theories of Kutateladze and Leontiev [15]. According to the present theory [equation (40)] this velocity ratio should equal the square root of the density ratio.

The effect of Prandtl and Schmidt number

In equations (7) through (11) it was shown how the simple model could be corrected to allow for the different rates of transfer of heat, mass and momentum. These corrections are only valid at low rates of mass transfer and it is of interest to determine what modifications are necessary if mass transfer is appreciable.

If one proceeds as in equations (14) and (15) there is an immediate problem about which of the velocities $(u_p)_o, (u_e)_o$ or $(u_c)_o$ to use when these are not all equal to u_o. Two possibilities are at once apparent:

(1) Use $(u_p)_o$ and modify the result by multiplying by $Pr^{-\frac{2}{3}}$ or $Sc^{-\frac{2}{3}}$ to get the heat- and mass-transfer coefficients.

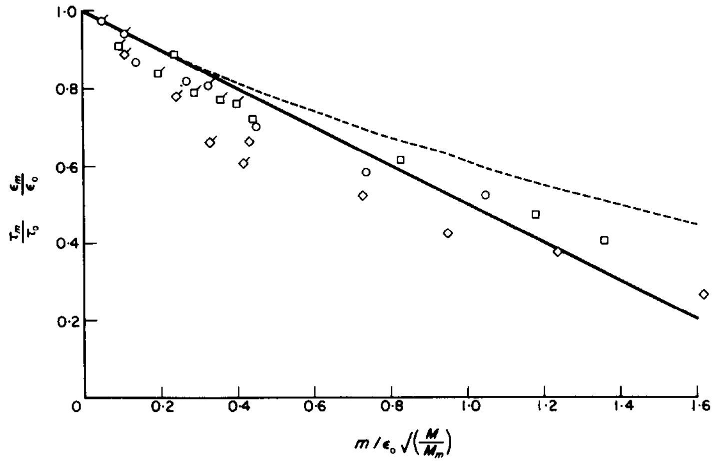

FIG. 5. Comparison of data of Pappas and Okuno with equations (44) and (45).

			Air	Freon-12	Helium
———	Equation (44)	Reynolds number 0·87 × 10^6	○	□	◇
-------	Equation (45)	4 × 10^6	○	□	◇

(2) Use the appropriate values of u_o from (8–10) throughout the analysis.

Following alternative (1) the result is

$$(\varepsilon_e)_m = \tfrac{1}{2}\rho[(u_m)_o - v]Pr^{-\frac{2}{3}} \tag{46}$$

$$(\varepsilon_c)_m = \tfrac{1}{2}\rho[(u_m)_o - v]Sc^{-\frac{2}{3}}. \tag{47}$$

Using (11) and (12) we then find

$$\frac{(\varepsilon_c)_m}{(\varepsilon_c)_o} = \frac{(\varepsilon_e)_m}{(\varepsilon_e)_o} = 1 - \frac{1}{2}\frac{m}{(\varepsilon_p)_o}. \tag{48}$$

Following the second alternative, on the other hand, we get

$$(\varepsilon_e)_m = \tfrac{1}{2}\rho[(u_e)_o - v] \tag{49}$$

$$(\varepsilon_c)_m = \tfrac{1}{2}\rho[(u_c)_o - v] \tag{50}$$

whence, using (11) and (12)

$$\frac{(\varepsilon_e)_m}{(\varepsilon_e)_o} = 1 - \frac{1}{2}\frac{m}{(\varepsilon_e)_o} = 1 - \frac{1}{2}\frac{m}{(\varepsilon_p)_o}Pr^{\frac{2}{3}} \tag{51}$$

$$\frac{(\varepsilon_c)_m}{(\varepsilon_c)_o} = 1 - \frac{1}{2}\frac{m}{(\varepsilon_c)_o} = 1 - \frac{1}{2}\frac{m}{(\varepsilon_p)_o}Sc^{\frac{2}{3}}. \tag{52}$$

A compromise between the above extreme views of the situation, represented by equations (48, 51, 52), is to adopt a logarithmic mean of the influence of Prandtl number and Schmidt number as follows,

$$\frac{(\varepsilon_e)_m}{(\varepsilon_e)_o} = 1 - \frac{1}{2}\frac{m}{(\varepsilon_e)_o}Pr^{-\frac{1}{3}} = 1 - \frac{1}{2}\frac{m}{(\varepsilon_p)_o}Pr^{\frac{1}{3}} \tag{53a}$$

$$\frac{(\varepsilon_c)_m}{(\varepsilon_c)_o} = 1 - \frac{1}{2}\frac{m}{(\varepsilon_c)_o}Sc^{-\frac{1}{3}} = 1 - \frac{1}{2}\frac{m}{(\varepsilon_p)_o}Sc^{\frac{1}{3}}. \tag{53b}$$

Equations (53a) and (53b) are compared with the solutions of Evans [17] for the laminar boundary layer on a flat plate in Fig. 7. In order to make a direct comparison the equations have been rearranged to the form

$$\frac{(\varepsilon_c)_m}{(\varepsilon_c)_o} = \frac{1}{1 + \frac{1}{2}[m/(\varepsilon_c)_m]\,Sc^{-\frac{1}{3}}}. \tag{54}$$

Method of using the theory

The above theory is used in essentially the same way as has been explained in detail by Spalding in his book [6]. Following Spalding's notation we consider three regions which define a "lumped-parameter" description of the problem. These are:

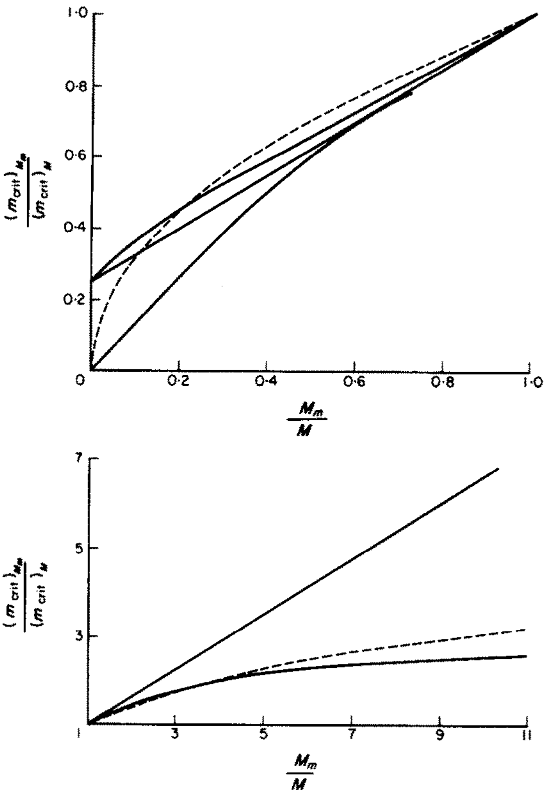

FIG. 6. Ratio of critical blowing rate for boundary-layer separation, $(m_{\text{crit}})_{M_m}$, to critical blowing rate with injection of gas of the same molecular weight as the main stream $(m_{\text{crit}})_M$, vs. molecular weight ratio M_m/M.

——— Various theories of Kutateladze and Leontiev [15];
------- Present theory.

The main flow. G state.
The region of fluid close to the wall. S state.
The interior of the wall (or adjacent fluid). T-state.

The mixing flux ε_m transfers energy, matter and momentum between the G and S states whereas the net flow m carries the same conserved properties from the T to the G state (or in the other direction). The S and T states are often considered to be in some form of two-phase equilibrium although this is not necessarily so (for example, a finite departure from equilibrium is necessary to cause high rates of boiling or condensation; injected gas may not be at the same temperature as the layer close to the wall). Conservation equations can be written for control volumes between the G and S, S and T, or G and T states. Detailed methods for defining the T state in complex situations are given by Spalding.

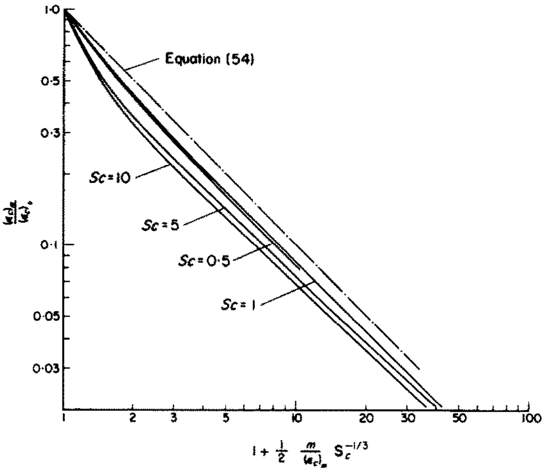

FIG. 7. Comparison between equation (54) and the solutions of Evans [17] for mass transfer in the laminar boundary layer.

Writing a conservation equation for the S layer for any conserved property X there follows

$$m(X_T - X_S)_m = \varepsilon_m(X_G - X_S)_{\varepsilon_m}. \qquad (55)$$

The subscripts on the brackets have been added to emphasize that the properties have to be evaluated for the ε_m and m streams which may be composed of completely different substances.

Equation (55) forms the basis for the definition by Spalding of a new parameter B, as follows

$$B = \frac{m}{\varepsilon_m} = \frac{(X_G - X_S)_{\varepsilon_m}}{(X_T - X_S)_m}. \qquad (56)$$

By considering a model in which differential transfer processes occur in series from the S to the G state, Spalding arrived at an expression for the influence of net mass transfer on the fluxes, as follows

$$\log\left(\frac{\varepsilon_2}{\varepsilon_1}\right) = \frac{m}{\varepsilon_o}. \tag{57}$$

This expression is symmetrical and is valid for both positive and negative values of m. Furthermore, it is consistent with the molecular diffusion laws for gases.

Comparison with the work of Kutateladze and Leontiev

Kutateladze and Leontiev [15] considered the effect of injection and suction on heat, mass and momentum transfer through turbulent boundary layers. Although the level of sophistication of their analysis is considerably greater than can be found in the present paper, many of their conclusions are not appreciably different. One comparison has already been made in Fig. 6, for example, A further interesting prediction by these authors is for the modification of wall shear stress due to injection of a gas with the same molecular weight as the main stream. With the present notation this becomes

$$\frac{\varepsilon_m}{\varepsilon_o} = \left(1 - \frac{m}{4\varepsilon_o}\right)^2 \left(1 + \frac{m}{4\varepsilon_o}\right)^{-\frac{1}{2}} \tag{58}$$

which is intermediate between the predictions of equations (27) and (29) over most of the range of positive values of m (Fig. 4).

Examples

The techniques for using the theory and a comparison between the results which the various equations predict are perhaps best illustrated by some examples, as follows:

(1) *The build-up of non-condensables near a surface on which condensation is occurring.* Suppose that condensation is occurring on a surface at a rate m'. Let the molal concentration of non-condensables in the main vapor stream be $(c_a)_s$ and the concentration at the wall be $(c_a)_w$.

Since there is no source or sink of non-condensables, the net fluxes to and from the wall must balance. Therefore, we have

$$\varepsilon_1 (c_a)_s = \varepsilon_2 (c_a)_w \tag{59}$$

whence, on rearrangement

$$\frac{(c_a)_w}{(c_a)_s} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_m + m'}{\varepsilon_m}. \tag{60}$$

Remembering that m' is the negative of m, we obtain, on substitution from (57, 16, 17) and (29), respectively

$$\frac{(c_a)_w}{(c_a)_s} = \exp(m/\varepsilon_o) \tag{61}$$

or

$$\frac{(c_a)_w}{(c_a)_s} = \frac{1 + \frac{1}{2} m/\varepsilon_o}{1 - \frac{1}{2} m/\varepsilon_o} \tag{62}$$

or

$$\frac{(c_a)_w}{(c_a)_s} = 1 + \frac{m}{\varepsilon_o} \exp(m/2\varepsilon_o). \tag{63}$$

These three equations can be compared by expanding them in turn as series in m/ε_o, as follows.

$$\exp(m/\varepsilon_o) = 1 + m/\varepsilon_o + \frac{1}{2}\left(\frac{m}{\varepsilon_o}\right)^2 + \frac{1}{6}\left(\frac{m}{\varepsilon_o}\right)^3 + \ldots \tag{64}$$

$$\frac{1 + \frac{1}{2} m/\varepsilon_o}{1 - \frac{1}{2} m/\varepsilon_o} = 1 + m/\varepsilon_o + \frac{1}{2}\left(\frac{m}{\varepsilon_o}\right)^2 + \frac{1}{4}\left(\frac{m}{\varepsilon_o}\right)^3 + \ldots \tag{65}$$

$$1 + \frac{m}{\varepsilon_o} \exp(m/2\varepsilon_o) = 1 + m/\varepsilon_o + \frac{1}{2}\left(\frac{m}{\varepsilon_o}\right)^2 + \frac{1}{8}\left(\frac{m}{\varepsilon_o}\right)^3 + \ldots. \tag{66}$$

The three series are identical as far as the term in $(m/\varepsilon_o)^2$ and are not very different in the cube term. A graphical comparison is also shown in Fig. 8.

(2) *Mass transfer from a surface into a flowing gas stream.* Consider mass transfer of component "a" into a gas stream which contains both components "a" and "b". Denote the overall pressure by P. The partial pressures of the components will add up to the total pressure everywhere, therefore,

$$P_a + P_b = P. \tag{67}$$

For small rates of mass transfer the mass flux from the surface will be

$$m = h_D \frac{M_a}{RT}[(P_a)_w - (P_a)_s] \tag{68}$$

if M_a is the molecular weight of a, R the universal gas constant, and subscripts w and s denote the wall and the stream as before.

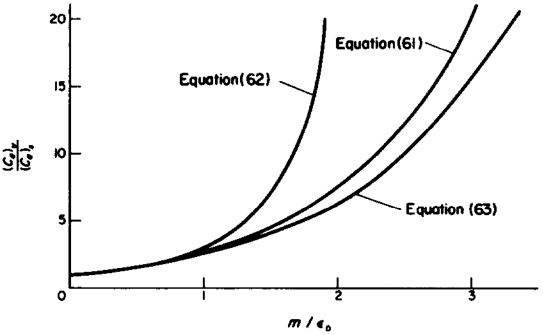

FIG. 8. Graphical comparison between equations (61–63) for predicting the accumulation of non-condensables close to the condensing surface.

Equation (68) can be expressed in terms of the transverse velocity model, as follows,

$$m = \frac{u_o}{2}\frac{M_a}{RT}[(P_a)_w - (P_a)_s]. \tag{69}$$

The superimposed net flow velocity from the wall is

$$v = m\frac{RT}{M_a P}. \tag{70}$$

Therefore, when v cannot be neglected in comparison with u_o, the mass transfer, which is the sum of the contributions from the stream approaching and the stream leaving the wall, becomes

$$m = \frac{M}{2RT}[(u_o + v)(P_a)_w - (u_o - v)(P_a)_s]. \tag{71}$$

Using equation (67) together with (70) and (71) we obtain

$$v[(P_b)_s + (P_b)_w] = u_o[(P_a)_w - (P_a)_s] \tag{72}$$

which is compatible with the condition that the overall flux of b should be zero, i.e.

$$(u_o + v)(P_b)_w = (u_o - v)(P_b)_s. \tag{73}$$

Eliminating v between (70) and (72) gives

$$m = u_o\frac{M_a}{RT}\cdot\frac{P}{(P_b)_s + (P_b)_w}[(P_a)_w - (P_a)_s]. \tag{74}$$

Comparing (74) with (69) it is clear that the effect of the net flow from the surface is to increase the mass transfer in the ratio

$$\frac{2P}{(P_b)_s + (P_b)_w}.$$

If the same problem is approached via equation (57) the analog of (73) is

$$\varepsilon_2(P_b)_w = \varepsilon_1(P_b)_s. \tag{75}$$

Use of (57) then gives

$$m/\varepsilon_o = \log\frac{(P_b)_s}{(P_b)_w}. \tag{76}$$

Interpreting m/ε_o as a ratio between molal fluxes so that

$$\varepsilon_o = h_D\frac{M_a}{RT}P \tag{77}$$

equation (76) becomes

$$m = h_D\frac{M_a}{RT}P\log\frac{(P_b)_s}{(P_b)_w}. \tag{78}$$

Compared with the prediction of equation (68), the mass transfer is therefore augmented by the factor

$$\frac{P}{(P_a)_w - (P_a)_s}\log\frac{(P_b)_s}{(P_b)_w}$$

which in view of (67), is the same as

$$\frac{P}{(P_b)_s - (P_b)_w} \log \frac{(P_b)_s}{(P_b)_w}.$$

This result can also be deduced from the laws of molecular diffusion [18, 19].

In order to relate the predictions of the two methods, we should compare the two ratios

$$\frac{2}{(P_b)_s + (P_b)_w}$$

and

$$\frac{1}{(P_b)_s - (P_b)_w} \log \frac{(P_b)_s}{(P_b)_w}.$$

Letting

$$\frac{(P_b)_s}{(P_b)_w} = \frac{1 + y}{1 - y},$$

it is sufficient to show that

$$\log \frac{1 + y}{1 - y} \approx 2y \qquad (79)$$

which is true up to terms of order y^3. For example, if $(P_b)_s = P$, $(P_b)_w = P/2$ we have $y = \frac{1}{3}$ and the error in equation (79) is less than 4 per cent.

Some further examples of the comparison between the different theories and experimental data are given in [14].

CONCLUSIONS

The equations, examples and graphical comparisons with experimental data which have been presented in this paper clearly demonstrate the power and utility of the Reynolds flux concept. This concept can evidently be extended far beyond the "Reynolds analogy" stage which is too often accorded merely cursory attention in current textbooks.

It is clear that the simplest model which leads to equations (16) and (17) is adequate for most practical purposes as long as the ratio m/ε_o is less than unity. At higher values of m/ε_o the accuracy of the theory deteriorates. However, this is just the point at which two-dimensional effects in the main flow become significant. This point will be discussed further in the next part of this paper.

The formulation in terms of transverse velocities is easier to deal with than the "mass flux" model when several components of different molecular weight are present. It is conceptually the simplest to imagine of the three alternative theories and this is a real virtue for teaching and presentation purposes. It is not intended to compete with the level of accuracy which can be achieved with more complex analytical techniques, such as boundary-layer theory, nevertheless it compares favorably with most of these more complex methods over a surprisingly wide range of conditions. It can be exploited in all of the manifold ways indicated by Spalding [6] as well as some additional ways which are shown in this and the subsequent paper.

The value of ε_o (or u_o) must usually be determined from a friction factor, heat-transfer or mass-transfer correlation which has been established for low values of m (or v). However, even when such a correlation is not available the theory provides a simple method for making general predictions and for organizing experimental results on a logical basis.

REFERENCES

1. O. Reynolds, On the extent and action of the heating surface of steam boilers, *Proc. Lit. Phil. Soc.*, Manchester **14**, 7–12 (1874).
2. W. Nusselt, Die Verbrennung und die Vergasung der Kohle auf dem Rost, *Z. Ver. Dr. Ing.* **60**, 102–107 (1916).
3. R. S. Silver, Application of the Reynolds analogy to the combustion of solid fuels, *Nature, Lond.* **165**, 725 (1950).
4. R. S. Silver, Theoretical treatment of combustion in fuel beds, *Fuel Lond.* **32**, 121–150 (1953).
5. R. S. Silver, Some personal views and experiences in heat transfer design, in *Proceedings of the Third International Heat Transfer Conference*. Am. Inst. Chem. Engrs, New York (1966).
6. D. B. Spalding, *Convective Mass Transfer*. McGraw-Hill, New York (1963).
7. R. S. Silver, An approach to the general theory of surface condensers, *Proc. Instn Mech. Engrs* **178**, 339–376 (1963–64).

8. G. B. Wallis, Pressure gradients for air flowing along porous tubes with uniform extraction at the walls, *Proc. Instn Mech. Engrs* **180**, 27–35 (1965–66).
9. R. S. Silver and G. B. Wallis, A simple theory for longitudinal pressure drop in the presence of lateral condensation, *Proc. Instn Mech. Engrs* **180**, 36–40 (1965–66).
10. T. H. Chilton and A. P. Colburn, Mass-transfer coefficients, prediction from data on heat transfer and fluid friction, *Ind. Engng Chem.* **26**, 1183–1187 (1934).
11. E. T. Jaynes, *Probability Theory in Science and Engineering*. McGraw-Hill, New York (1958).
12. M. Tribus, *Thermostatics and Thermodynamics*. Van Nostrand, Princeton, N.J. (1961).
13. R. Iglisch, Exact calculation of laminar boundary layer in longitudinal flow over a flat plate with homogeneous suction, NACA Tech. Memo. 1205, p. 68 (1949).
14. G. B. Wallis, Discussion of Paper 81, Session 9, in *Proceedings of the Third International Heat Transfer Conference*, Chicago, Illinois. Am. Inst. Chem. Engrs, New York (1966).
15. S. S. Kutateladze and A. I. Leontiev, *Turbulent Boundary Layers in Compressible Gases*, translated by D. B. Spalding. Academic Press, New York (1964).
16. C. C. Pappas and A. F. Okuno, Measurements of skin friction of the compressible turbulent boundary layer on a cone with foreign gas injection, *J. Aero/Space Sci.* **27**, 321–333 (1960).
17. H. L. Evans, Mass transfer through laminar boundary layers—6, *Int. J. Heat Mass Transfer* **3**, 321–339 (1961).
18. R. B. Bird, W. E. Stewart and E. N. Lightfoot, *Transport Phenomena*, 3rd edn., pp. 658–666. John Wiley, New York (1963).
19. W. M. Rohsenow and H. Choi, *Heat, Mass and Momentum Transfer*, p. 400. Prentice-Hall, New Jersey (1961).
20. J. P. Hartnett and E. R. G. Eckert, Mass-transfer cooling in a laminar boundary layer with constant fluid properties, *Trans. Am. Soc. Mech. Engrs* **79**, 247–254 (1957).
21. K. Torii, N. Nishiwaki and M. Hirata, Heat transfer and skin friction in turbulent boundary layer with mass injection, in *Proceedings of the Third International Heat Transfer Conference*, Paper No. 85, Vol. 3, pp. 34–48. Am. Inst. Chem. Engrs, New York (1966).
22. R. M. Olson and E. R. G. Eckert, Experimental studies of turbulent flow in a porous circular tube with uniform injection through the tube wall, *J. Appl. Mech.* **33**, 7–17 (1966).

Résumé—Cette série de deux articles a l'objectif suivant: (1) Attirer l'attention sur l'utilité du concept du flux de Reynolds dans l'analyse à but technique, pour comparer les différentes théories deschercheurs antérieurs et pour présenter quelques developpements nouveaux. (2) Montrer comment le concept peut être appliqué pour prévoir les gradients de pression dans l'ébullition par convection forcée, l'ébullition éclair et la condensation. Cette analyse est également reliée au mécanisme de production d'entropie dans l'écoulement diphasique.

Zusammenfassung—Diese aus zwei Arbeiten bestehende Abhandlung hat folgendes Ziel: (1) Die Nützlichkeit der Reynold'schen Flussüberlegung für die Ingenieuranalysis darzulegen, um verschiedene Theorien früherer Untersuchungen zu vergleichen und einige neuere Entwicklungen anzugeben. (2) Die Überlegung auf die Berechnung von Druckgradienten beim Sieden unter Zwangskonvektion, bei Versprühen und Kondensation anzuwenden. Diese Analyse bezieht sich auch auf den Mechanismus der Entropievermehrung in der Zweiphasenströmung.

Аннотация—Данная работа, состоящая из двух статей проделана с целью : (1) привлечь внимание к целесообразности применения понятия рейнольдсова потока для инженерных расчетов, сравнить различные теории других исследователей, а также предложить новые разработки ; (2) показать, как это понятие можно применить к расчету градиентов давления при кипении при вынуждеиной конвекции, мгновенном испарении и конденсации.

Данный анализ также связан с механизмом производства энтропии в двухфазном потоке.